We assign to each side of a cube a number from 1 to 12 in a one - to - one correspondence. Prove that it is not possible to do it in such a way that the sum of the numbers assigned to the edges going into the same vertex is the same for every vertex.

Deadline for solution: 11/16/09. Send solution at gprajitu@brockport.edu or drop hard copy in Dr. Prajitura's mail box.

Last week's problem:
If $p > q$ are odd prime numbers such that

$$n = \frac{p^2 + q^2}{p - q}$$

is an integer, prove that $n - 1$ is a square.

If n is an integer then $p - q | p^2 + q^2$. Since obviously $p - q | p^2 - q^2$, we get that

$$p - q | 2p^2 = (p^2 + q^2) + (p^2 - q^2).$$

Then there is k such that $2p^2 = k(p - q) \iff p(k - 2p) = kq$ which implies that $p|kq$. Since $\gcd(p, q) = 1$, we conclude that $p|k$. Therefore there is a such that $k = pa$ and so, from the previous equation, we get that $p(pa - 2p) = paq \iff pa - 2p = aq$ which implies that $p|aq$. Since $\gcd(p, q) = 1$, we conclude that $p|a$. Therefore there is m such that $a = pm$ and so, from the previous equation, we get that $p^2 m - 2p = pmq \iff m(p - q) = 2$. Since $p - q \geq 2$, we get that $p - q = 2$ and $m = 1$. Therefore $p = q + 2$ and thus

$$n - 1 = \frac{(q + 2)^2 + q^2}{2} - 1 = \frac{2q^2 + 4q + 4}{2} - 1 = q^2 + 2q + 1 = (q + 1)^2.$$

No solution was received for this problem.

Problem for graduate students:
Solve the equation

$$2(x^2 - x + 3) + 2(y^2 - y + 1) = 7$$

Last week's problem:
If the equation

$$x^3 + (a + 1)x + b = 0$$

has only integer solutions, none of them equal to 0, prove that $a^2 + b^2$ cannot be a prime number.

Let x_1, x_2, x_3 be the solutions of the equation. From Vieta's relations we get that

$$\begin{cases} x_1 + x_2 + x_3 & = 0 \\ x_1x_2 + x_2x_3 + x_3x_1 & = a + 1 \\ x_1x_2x_3 & = -b \end{cases}$$

From the first equation we get that $x_3 = -(x_1 + x_2)$. Substituting it in the other two equations we get that

$$x_1x_2 + x_3(x_1 + x_2) = a + 1 \iff x_1x_2 - (x_1 + x_2)^2 = a + 1$$

$$\iff x_1x_2 - (x_1 + x_2)^2 - 1 = -a$$

and

$$x_1x_2(x_1 + x_2) = b$$

Therefore

$$a^2 + b^2 = (x_1x_2 - (x_1 + x_2)^2 - 1)^2 + x_1^2x_2^2(x_1 + x_2)^2$$

$$= x_1^2x_2^2 + (x_1 + x_2)^4 + 1 - 2x_1x_2(x_1 + x_2)^2 - 2x_1x_2 + 2(x_1 + x_2)^2 + x_1^2x_2^2(x_1 + x_2)^2$$

$$= (1 + (x_1 + x_2)^2)(x_1^2x_2^2 - 2x_1x_2 + 1 + (x_1 + x_2)^2)$$

$$= (1 + (x_1 + x_2)^2)((x_1x_2 - 1)^2 + (x_1 + x_2)^2)$$

Suppose that $a^2 + b^2$ is a prime number. Then one of the factors in the decomposition above must be 1.

If $1 + (x_1 + x_2)^2 = 1$ then $(x_1 + x_2)^2 = 0$ and thus $x_1 + x_2 = 0$ which means that $x_2 = -x_1$. Then

$$a^2 + b^2 = (x_1x_2 - 1)^2 = (-x_1^2 - 1)^2 = (x_1^2 + 1)^2$$

Since $a^2 + b^2$ is a prime numebr, $x_1^2 + 1 = 1$ which implies that $x_1 = 0$, a contradiction.

Therefore the second factor must be 1. This means that $(x_1x_2 - 1)^2 + (x_1 + x_2)^2 = 1$. The sum of two positive integers equals 1. The only way to have something like this is if one of them is 1 and the other one is 0.

If $(x_1x_2 - 1)^2 = 1$ and $(x_1 + x_2)^2 = 0$ we get, as above $x_1 + x_2 = 0 \iff x_2 = -x_1$. Therefore $1 = (x_1x_2 - 1)^2 = (-x_1^2 - 1)^2$ and as above this implies that $x_1 = 0$, a contradiction.

If $(x_1x_2 - 1)^2 = 0$ and $(x_1 + x_2)^2 = 1$ we get, from the first equation, that $x_1x_2 = 1$. Therefore $x_1 = x_2 = 1$ or $x_1 = x_2 = -1$. Using this in the second equation we get that $4 = 1$, a contradiction.

Thus $a^2 + b^2$ cannot be a prime.

Correct solution was submitted by Darryl George who also solved the problem from the previous week.